A catalogue of complete group presentations

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Rapports de Recherche

N° 439

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Août 1985

RESUME

Une présentation complète d'un groupe est formée d'un ensemble de générateurs d'un ensemble de règles de récriture engendrant une ralation bien-fondée et confluente sur les mots, résolvant ainsi le problème du mot pour cette présentation. Des présentations complètes pour les groupe de surfaces, de Coxeter, ainsi que pour les groupes polyhédraux et symétriques sont données. Elles fournissent des algorithmes uniformes et efficaces pur le problème du mot dans ces groupes.

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ABSTRACT

A complete group presentation is defined by a set of generators and a set of replacement rules generating a well-founded and confluent relation on words, thereby solving the word problem for this presentation. Complete presentations for Surface, Coxeter, Polyhedral and Symmetric groups are given. These complete presentations possess interesting combinatorial properties, they provide uniform and efficient algorithms for the word problem, and succinctly describe Cayley graphs.

1. Introduction

This paper reports experiments on group presentations of an implementation of the Knuth-Bendix completion procedure. The techniques of rewriting systems, classical in computer science [Knu70, LeC85], have been investigated in groups by several authors [Gil79, Büc79, Bau81, LeC83]. It was of interest to check whether or not some usual groups were tractable by such means. The answer seems to be affirmative as we try to show in this catalogue paper. For practical efficiency, the catalogue length must not hide the complexity of completion. Bauer [Bau81] has shown that every complexity class can be coded by word rewriting systems. However, this algorithm must be though as a compilation process, by opposition to the Todd-Coxeter [Tod36] enumeration technique for example. Moreover, some complete presentations encode efficient word problem algorithms for parametrized group presentations, possibly including infinite sets of rules.

The presentations are taken from [Cox72]. The complete presentations were intuited with a LISP implementation of the completion procedure [LeC84]. The reader will find complete presentations of:

- The fundamental groups of p-holes torus and projective planes. They initiated Dehn's study of small cancellation. The complete presentations have nice combinatorial properties designing an efficient word problem algorithm.
- The Coxeter groups, discrete transformation groups generated by reflections through hyperplanes. Here, the partial commutativity of some generators is a case of failure. More interesting is the fact that this class possesses infinite sets of rules. Moreover a single parametrized rule describes the complete presentation.
- The polyhedral groups, generated by rotations in the spherical or hyperbolic geometry, paving the whole sphere or half-plane by non-regular polygons. They are the rotation subgroups of Coxeter groups. Length-increasing rules appear in some complete presentations, the termination remains open in such cases.

- The symmetric groups S_n, several complete systems exist, depending on the given generators. They also have nice properties. Generally these sytems have a great number of rules (sometimes $n!$, the size of S_n), but far from the theoretical upper bound ($2|G|\times|\mathbf{G}|$ for a group **G** with $|G|$ generators [LeC85]). As ususal, the rules possess a terse description. The complete presentations are closely related to some sorting algorithms.

Let us now briefly present the notion of complete presentation. Let G be a set of generators. A ***rule*** is a pair of words (u,v) in G^*, noted $u \to v$. Given two rules $ua \to v$ and $au' \to v'$, $a \in G^*$, $a \neq 1$, we say that they ***superpose*** on a, the word uau' reduces on vu' and uv'. This last pair of words is called a ***critical pair***. A set R of rules defines on G^* a reduction relation noted $\overset{*}{\underset{R}{\Rightarrow}}$, which is the reflexive-transitive closure of the relation $w \underset{R}{\Rightarrow} w'$ iff $w=aub$ and $w'=avb$, $u \to v \in R$.

- The set of rules R is noetherian iff the relation $\underset{R}{\Rightarrow}$ is well-founded.
- The relation $\underset{R}{\Rightarrow}$ is confluent iff
  $\forall u,v,v' \in G^*,\ u \underset{R}{\Rightarrow} v$ and $u \underset{R}{\Rightarrow} v' \Rightarrow \exists w \in G^*\cdot v \overset{*}{\underset{R}{\Rightarrow}} w$ and $v' \overset{*}{\underset{R}{\Rightarrow}} w$.

Especially, a critical pair is said resolved or confluent when there exists such a word w. Noetherianity of R is usually proved by well-founded partial orderings such that:

- $\forall u \to v \in R,\ u > v$.
- $\forall a,b,u,v \in G^*,\ u > v \Rightarrow aub > avb$.

Let us call such ordering ***reduction orderings***. The confluence is checked by the following theorem [Knu70,Hue80]:

Theorem 1 (Knuth-Bendix)
The noetherian relation R is confluent iff its critical pairs are resolved.

Therefore, in presence of a finite presentation, we can orientate the defining equations according to a reduction ordering. If the resulting noetherian set of rules is not confluent, just add to the rules an unresolved critical pair. This is the essence of the completion algorithm [LeC85]. A noetherian and confluent set of rules is called a ***complete presentation***. It is easily seen that when the number of rules is finite, the set of normal forms is regular. As we deal with groups, implicit in all complete presentations is the set $F_G=\{aa^{-1} \to 1, a^{-1}a \to 1 \mid a \in G\}$. Reductions by this set will be noted $\underset{F}{\Rightarrow}$. If we resolve only critical pairs between a rule in F_G and other ones and if the rules do not increase the word length, we roughly obtain Dehn algorithm, see [LeC85]. This restricted completion will be called symmetrization of a presentation.

To establish noetherianity of a system, we use two classical orderings. The lexicographic one, abbreviated LEX, is defined as usual first by the length of the words, then at equal length by a total ordering on generators. This ordering can be refined by a weight on the generators, the weight of a word being the sum of its generators. At equal length a total ordering on generators discriminates the words.
For both orderings, we have $\forall u,v,a,b \in G^*,\ u > v \Rightarrow aub > avb$.

2. Surface Groups

2.1. Orientable Surfaces

The defining presentation of a p holes torus is

$$T_p = (A_1, \ldots, A_{2p}; A_1 \cdots A_{2p} = A_{2p} \cdots A_1).$$

The pieces of the symmetrized set of relations are just the generators and their inverses, so that when $p \geq 2$, the presentation has a word problem solved by Dehn algorithm. Surprisingly, the Knuth-Bendix completion shows that Dehn algorithm not only insures the confluence on the unit element 1, but also, when expressed as a set of rules, gives a normal form for each element, as the symmetrized systems are also complete.

The case $p = 1$ defines the group $\mathbf{Z} \times \mathbf{Z}$. Its complete presentation is $\{BA \rightarrow AB, B^{-1}A^{-1} \rightarrow A^{-1}B^{-1}, BA^{-1} \rightarrow A^{-1}B, B^{-1}A \rightarrow AB^{-1}\}$. The following case $p=2$ gives the complete system:

$$T_2 \left\{ \begin{array}{cc} DCBA \rightarrow ABCD & D^{-1}C^{-1}B^{-1}A^{-1} \rightarrow A^{-1}B^{-1}C^{-1}D^{-1} \\ BCDA^{-1} \rightarrow A^{-1}DCB & B^{-1}C^{-1}D^{-1}A \rightarrow AD^{-1}C^{-1}B^{-1} \\ B^{-1}A^{-1}DC \rightarrow CDA^{-1}B^{-1} & BAD^{-1}C^{-1} \rightarrow C^{-1}D^{-1}AB \\ DA^{-1}B^{-1}C^{-1} \rightarrow C^{-1}B^{-1}A^{-1}D & D^{-1}ABC \rightarrow CBAD^{-1} \end{array} \right.$$

The second observation about this system, after the fact that it is both symmetrized and complete, is that all rules have the form $\lambda \rightarrow \bar{\lambda}$, where $\bar{\lambda}$ is the reverse word of λ. In the general case, the completion gives a system T_p of $4p$ rules composed of words whose length is $2p$.

$$\begin{array}{ccc} A_{2k} \cdots A_{2p} A_1^{-1} \cdots A_{2k-1}^{-1} & \rightarrow & A_{2k-1}^{-1} \cdots A_1^{-1} A_{2p} \cdots A_{2k} \\ A_{2k} \cdots A_1 A_{2p}^{-1} \cdots A_{2k+1}^{-1} & \rightarrow & A_{2k+1}^{-1} \cdots A_{2p}^{-1} A_1 \cdots A_{2k} \\ A_{2k}^{-1} \cdots A_1^{-1} A_{2p} \cdots A_{2k+1} & \rightarrow & A_{2k+1} \cdots A_{2p} A_1^{-1} \cdots A_{2k}^{-1} \\ A_{2k}^{-1} \cdots A_{2p}^{-1} A_1 \cdots A_{2k-1} & \rightarrow & A_{2k-1} \cdots A_1 A_{2p}^{-1} \cdots A_{2k}^{-1} \end{array}$$

where $k = 1, \ldots, p$. The proof that a system of rules defines a complete presentation requires three steps: the termination, the rules are consequences of the definition, and all critical pairs are resolved. The termination is proved by a Lex ordering such that
$A_{2p} > A_{2p}^{-1} > A_{2p-2} > A_{2p-2}^{-1} > \cdots > A_2 > A_2^{-1} > A_1 > A_1^{-1} > A_3 > A_3^{-1} > \cdots > A_{2p-1} > A_{2p-1}^{-1}$.
For each rule $\lambda \rightarrow \rho$, the word $\lambda\rho^{-1}$ is a cyclic permutation of the defining relation or its inverse. The last part has been mechanically checked.

Such rewriting sets define three algorithms, one for reducing an arbitrary word to its normal form, and two others performing the two group operations on normal forms: multiplication and inverses computation. Of course a solution to the former operation must use the fact that reductions may occur only at the joint of the two initial words. Let us mention a first insight in the computation of normal forms by giving an upper bound to the number of T_p-reductions. Book [Boo82] has proved that whithout length-increasing rules, a rewriting system on words possesses a linear-time reduction algorithm. For torus groups, this result is strengthened into an algorithm that does not perform backward searching. If we do not distinguish between the generators and their inverses, rules split in two distinct types according to wether or not the indices are increasing. Let the letter a_k denote either the generator A_k or its inverse, with the obvious meaning for a_k^{-1}. Only even letters a_{2k} appear as first letter in rules left members. Let $M = Wa_{2k} \cdots a_{2k+1}^{-1} W'$ be a word whose leftmost T_p-redex is the one displayed (the other type of reduction ending with a_{2k-1} is similar). We assume that Wa_{2k} is F-reduced, then $M \rightarrow Wa_{2k+1}^{-1} \cdots a_{2k} W'$. What are the eventual F or T_p-rules reducing a suffix of W? We have two cases:

1) F-reduction. This reduction is $a_{2k+1}a_{2k+1}^{-1} \rightarrow 1$, and $M \xrightarrow{*} W_0 a_{2k+2}^{-1} a_{2k+3}^{-1} \cdots a_{2k-1} a_{2k}$. No further reduction is then possible. For a F-redex implies $W=W_0 a_{2k+1}=W_1 a_{2k+2} a_{2k+1}$ and the initial T_p-redex would not be the leftmost one. Any T_p-redex implies $W=W_0 a_{2k+1}=W_1 a_{2k}^{-1} a_{2k+1}^{-1} a_{2k+1}$ and the subword preceding the initial redex would not be F-irreducible.

2) T_p-reduction. Every T_p-redex using a W suffix either must include the letter a_{2k}^{-1}, but this is impossible as the word $Wa_{2k}=W_0 a_{2k}^{-1} a_{2k}$ would be F-irreducible, or this new redex inverses its slope just on the joint of W and the initial redex, that is $W=W_1 a_{2j} a_{2j+1} \cdots a_{2p} a_1^{-1} \ldots a_{2p}^{-1}$, but once more the initial word would be F-reducible by $a_{2p} a_{2p}^{-1} \rightarrow 1$.

The leftmost redex strategy needs at most one F-reduction backwards after a T_p-reduction. Now, what is the next index to look at for the remaining reduction? If the next T_p-redex has a common subword with the right member just introduced, we are in the following case:

$$a_{2k+1}^{-1} \cdots a_{2j}^{-1} \cdots a_{2p}^{-1} a_1 \cdots a_{2k} a_{2k+1} \cdots a_{2j-1} W'_0 \text{ with } a_{2k+1} \cdots a_{2j-1} W'_0 = W'$$

but this implies that M would be F-reducible at the joint of W' and the initial redex. Thus, we have to T_p-reduce the leftmost redex only if we cannot F-reduce on both sides of this T_p-redex. Free cancellations have higher priority. The case is similar when the inversion of the redex slope occurs at the junction (cf. previous analysis). Thus we search left members in W' . But F-reductions are possible after the initial reduction with W' , this leads to a special case where the new right member disappears entirely:

$$M=Wa_{2k} \cdots a_1 a_{2p}^{-1} \cdots a_{2k+1}^{-1} a_{2k}^{-1} a_{2k-1}^{-1} \cdots a_1^{-1} a_{2p} \cdots a_{2k+1} W'_0 \rightarrow WW'_0$$

and in this case we must move backwards in W of $2p$-1 letters to restart the reductions. However, the previous special case can be checked easily, and it reduces the length of M by $4p$ letters, moving backwards of $2p$-1 after this is then as going forward by $2p$+1. So that in any case we have at most $|M|/2p$ T_p-reductions to reach the normal form of M. This analysis sketches the reduction algorithm.

The group $(A,B,C;ABC=CBA)$ has a symmetrized set that is also canonical.

$$\perp_1 \left\{ \begin{array}{lll} CBA \rightarrow ABC & & A^{-1}B^{-1}C^{-1} \rightarrow C^{-1}B^{-1}A^{-1} \\ BCA^{-1} \rightarrow A^{-1}CB & & AC^{-1}B^{-1} \rightarrow B^{-1}C^{-1}A \\ B^{-1}A^{-1}C \rightarrow CA^{-1}B^{-1} & & C^{-1}AB \rightarrow BAC^{-1} \end{array} \right.$$

But its noetherianity does not follow from a classical ordering. It belongs to the family $\perp_p$ defined by $(A_1, \ldots, A_{2p+1}; A_1 \cdots A_{2p+1}=A_{2p+1} \cdots A_1)$. Here is a complete presentation $\perp_p$ having $4p$+2 rules, the words length is $2p$+1.

$$A_{2k+1} \cdots A_1 A_{2p+1}^{-1} \cdots A_{2k+2}^{-1} \rightarrow A_{2k+2}^{-1} \cdots A_{2p+1}^{-1} A_1 \cdots A_{2k+1}$$

$$A_{2k+1}^{-1} \cdots A_{2p+1}^{-1} A_1 \cdots A_{2k} \rightarrow A_{2k} \cdots A_1 A_{2p+1}^{-1} \cdots A_{2k+1}^{-1}$$

$$A_{2k} \cdots A_{2p+1} A_1^{-1} \cdots A_{2k-1}^{-1} \rightarrow A_{2k-1}^{-1} \cdots A_1^{-1} A_{2p+1} \cdots A_{2k}$$

$$A_{2k}^{-1} \cdots A_1^{-1} A_{2p+1} \cdots A_{2k+1} \rightarrow A_{2k+1} \cdots A_{2p+1} A_1^{-1} \cdots A_{2k}^{-1}$$

where k=0, . . . ,p. The noetherianity of $\perp_p$ follows from the fact that each member of $G \cup G^{-1}$ appears in one an only one rule as prefix. All rules having the same length, we may restrict our attention to reduction chains of words of the same length.

Lemma 2

Let U and V be two words of the same length, and $b_1 \cdots b_{2p}$ be a left member prefix, then no reduction $b_1 \cdots b_{2p} U \xrightarrow[\perp_p]{*} b_{2p}^{-1} \cdots b_1^{-1} V$ can occur.

Proof. The proof is by induction on $|U|$. The proposition is trivial for both $|U|=0$ and $|U|=1$ from left members irreducibility. Set $P=b_1 \cdots b_{2p}=A_{2p+1} \cdots A_2$, due to the symmetry of the rules other cases are similar. Any reduction from PU to $P^{-1}V$ uses the first rule $A_{2p+1} \cdots A_1 \rightarrow A_1 \cdots A_{2p+1}$ as $b_1=A_{2p+1}$ must be reduced and this is the only rule starting with this letter (let us recall that reductions preserve length, we cannot have $A_{2p+1}A_{2p+1}^{-1} \rightarrow 1$). Then we have the following sequence of reductions:

$$\begin{aligned} b_1 \cdots b_{2p} U &\xrightarrow{*} A_{2p+1} \cdots A_2 A_1 U' \\ &\rightarrow A_1 A_2 \cdots A_{2p+1} U' \end{aligned}$$

Afterwards, A_2^{-1} at index 1 is possible only when A_1 disappears, which in turn is possible only by rule $A_1 A_{2p+1}^{-1} \cdots A_2^{-1} \rightarrow A_2^{-1} \cdots A_{2p+1}^{-1} A_1$, thus we have necessarily $A_2 \cdots A_{2p+1} U' \xrightarrow{*} A_{2p+1}^{-1} \cdots A_2^{-1} V'$, which contradicts the induction hypothesis ∎

Corollary 3

Let U and V be such that $|U|=|V|$, then one never has $\rho_i U \xrightarrow[\perp_p]{*} \lambda_j V$, ρ_i (resp. λ_j) right member of a rule (resp. left).

Thus the reductions must halt as a prefix may be reduced only once time and length never increases. Geometrically, the two families of complete presentations possess a terse description with $4p$ and $4p+2$ gons. We give them for T_2 and $\perp_1$:

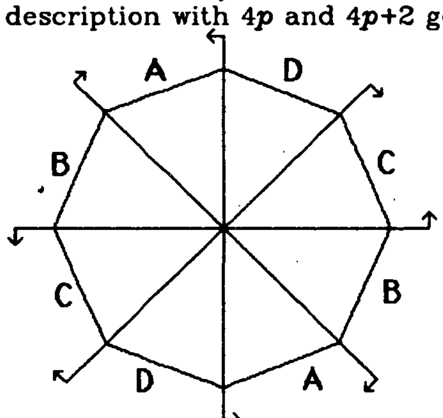

Fig. 1, T_2

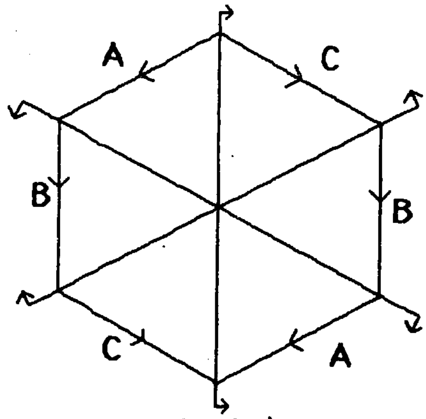

Fig. 2, $\perp_1$

The polygons represent an elementary circuit in the Cayley diagram of the groups. Two paths start from a given vertex to the opposite one. The arrows show irreducible paths. All rules are coded on these graphs.

2.2. Non-Orientable Surfaces

The non-orientable surface groups are defined by $(A_1, \ldots, A_p \,;\, A_1^2 \cdots A_p^2 = 1)$. There are two cases: p odd or even. We first give the two sets R_3 and R_4:

$$\left\{\begin{array}{rrr}
B^{-2}A^{-1} \to CCA & CCAA \to B^{-2} & A^{-2}CAA \to BBCB^{-2} \\
A^{-1}C^{-2} \to ABB & BBCC \to A^{-2} & A^{-1}C^{-1}AAB \to ABBC^{-1}B^{-1} \\
C^{-2}B^{-1} \to AAB & CAAB \to C^{-1}B^{-1} & B^{-1}A^{-1}BBC \to BCCA^{-1}C^{-1} \\
B^{-1}A^{-2} \to BCC & AABB \to C^{-2} & B^{-2}ABB \to CCAC^{-2} \\
A^{-2}C^{-1} \to BBC & ABBC \to A^{-1}C^{-1} & C^{-1}B^{-1}CCA \to CAAB^{-1}A^{-1} \\
C^{-1}B^{-2} \to CAA & BCCA \to B^{-1}A^{-1} & C^{-2}BCC \to AABA^{-2}
\end{array}\right.$$

$$\left\{\begin{array}{rrr}
A^{-2}D^{-2} \to BBCC & CDDAA \to C^{-1}B^{-2} & B^{-1}A^{-2}DAAB \to BCCDC^{-2}B^{-1} \\
C^{-1}B^{-2}A^{-1} \to CDDA & DDAAB \to C^{-2}B^{-1} & A^{-2}D^{-1}AABB \to BBCCD^{-1}C^{-2} \\
A^{-1}D^{-2}C^{-1} \to ABBC & BCCDD \to B^{-1}A^{-2} & C^{-1}B^{-2}ABBC \to CDDAD^{-2}C^{-1} \\
D^{-2}C^{-2} \to AABB & DAABB \to D^{-1}C^{-2} & A^{-1}D^{-2}CDDA \to ABBCB^{-2}A^{-1} \\
B^{-2}A^{-2} \to CCDD & AABBC \to D^{-2}C^{-1} & B^{-2}A^{-1}BBCC \to CCDDA^{-1}D^{-2} \\
B^{-1}A^{-2}D^{-1} \to BCCD & ABBCC \to A^{-1}D^{-2} & D^{-2}C^{-1}DDAA \to AABBC^{-1}B^{-2} \\
D^{-1}C^{-2}B^{-1} \to DAAB & CCDDA \to B^{-2}A^{-1} & D^{-1}C^{-2}BCCD \to DAABA^{-2}D^{-1} \\
C^{-2}B^{-2} \to DDAA & BBCCD \to A^{-2}D^{-1} & C^{-2}B^{-1}CCDD \to DDAAB^{-1}A^{-2}
\end{array}\right.$$

The general complete set of rules R_p depends on the parity of p. Let $n = [p/2]$, $A_i = A_{i+p}$ if $i = 1-p, \ldots, 0$ and $A_i = A_{i-p}$ if $i = p+1, \ldots, 2p$. Note that $A_{k-n} = A_{k+n+1}$. Both cases split in three sets of rules:

$$p = 2n+1$$

$$A_k^{-1}A_{k-1}^{-2} \cdots A_{k-n}^{-2} \quad \to \quad A_k A_{k+1}^2 \cdots A_{k+n}^2$$

$$A_k^{-2}A_{k-1}^{-2} \cdots A_{k-n+1}^{-2}A_{k-n}^{-1} \quad \to \quad A_{k+1}^2 \cdots A_{k+n}^2 A_{k+n+1}$$

$$A_k^2 \cdots A_{k+n}^2 \quad \to \quad A_{k-1}^{-2} \cdots A_{k-n}^{-2}$$

$$A_k A_{k+1}^2 \cdots A_{k+n}^2 A_{k+n+1} \quad \to \quad A_k^{-1}A_{k-1}^{-2} \cdots A_{k-n+1}^{-2}A_{k-n}^{-1}$$

$$A_k^{-2} \ldots A_{k-n+1}^{-2} A_{k-n} A_{k-n+1}^{2} \ldots A_k^2 \longrightarrow A_{k+1}^2 \ldots A_{k+n}^2 A_{k+n+1} A_{k+n}^{-2} \ldots A_{k+1}^{-2}$$

$$A_k^{-1} A_{k-1}^{-2} \ldots A_{k-n+1}^{-2} A_{k-n}^{-1} A_{k-n+1}^{2} \ldots A_k^2 A_{k+1} \longrightarrow A_k A_{k+1}^2 \ldots A_{k+n}^2 A_{k+n-1}^{-1} A_{k+n}^{-2} \ldots A_{k+2}^{-2} A_{k+1}^{-1}$$

$$n=2p$$

$$A_k^{-2} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} \longrightarrow A_{k+1}^2 \cdots A_{k+n}^2$$

$$A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n}^{-1} \longrightarrow A_k A_{k+1}^2 \cdots A_{k+n-1}^2 A_{k+n}$$

$$A_k^2 \cdots A_{k+n-1}^2 A_{k+n} \longrightarrow A_{k-1}^{-2} \cdots A_{k-n+1}^{-2} A_{k-n}^{-1}$$

$$A_k A_{k+1}^2 \cdots A_{k+n}^2 \longrightarrow A_k^{-1} A_{k-1}^{-2} \cdots A_{k-n+1}^{-2}$$

$$A_k^{-2} \ldots A_{k-n+2}^{-2} A_{k-n+1} A_{k-n+2}^{2} \ldots A_{k+1}^2 \longrightarrow A_{k+1}^2 \ldots A_{k+n}^2 A_{k+n+1}^{-1} A_{k+n}^{-2} \ldots A_{k+2}^{-2}$$

$$A_k^{-1} A_{k-1}^{-2} \ldots A_{k-n+1}^{-2} A_{k-n} A_{k-n+1}^{2} \ldots A_{k-1}^2 A_k \longrightarrow A_k A_{k+1}^2 \ldots A_{k+n-1}^2 A_{k+n} A_{k+n-1}^{-2} \ldots A_{k+1}^{-2} A_k^{-1}$$

For all these rules, k ranges from 1 to p.
Termination: Lex Ordering with $A_1^{-1} > \cdots > A_p^{-1} > A_1 > \cdots > A_p$.
Every set has $6p$ rules. In both cases, the first four rules are symmetrized presentations, while the last two rules arise from critical pairs with the pieces $A_i \in G \cup G^{-1}$. For example, if $p=4$ the two rules

$$A_1A_1A_2A_2A_3 \longrightarrow A_4^{-1}A_4^{-1}A_3^{-1}$$
$$A_3A_4A_4A_1A_1 \longrightarrow A_3^{-1}A_2^{-1}A_2^{-1}$$

superposed on the piece A_1 creates the new rule:

$$A_3^{-1}A_2^{-1}A_2^{-1}A_1A_2A_2A_3 \longrightarrow A_3A_4A_4A_1A_4^{-1}A_4^{-1}A_3^{-1}.$$

3. Coxeter Groups

3.1. The Completion

The word problem for Coxeter groups has been proved decidable by Tits [Tit69] with an algorithm enumerating the finite set of derivatives of a word under a relation generated by a finite set of rules. This reduction is confluent but not noetherian, more precisely, if $\lambda \longrightarrow \rho$ is a rule, then $\rho \longrightarrow \lambda$ also. The completion strengthens this relation into a noetherian one. However, this new relation does not handle groups having commuting pairs of generators.

The completion of these group presentations is perhaps the more convincing example of the power of rewriting systems. By elementary combinatorial methods, it proves the solvability of the word problem whereas geometrical methods are usually used [Bou78]. Furthermore, the family is parametrized by $n \times n$ symmetric integer matrices. Despite this high number of parameters, a terse description of complete presentations is found that leads to an efficient word problem algorithm, as for torus groups. However, a drawback is encountered. The partial commutativity of a presentation leads to a failure.

Let I be a finite set of n generators. A Coxeter matrix on I is a function $M: I \times I \longrightarrow \mathbb{N} \cup \{\infty\}$ such that for all i,j in I, $M(i,i)=1$ and $M(i,j)=M(j,i) \geq 2$ if $i \neq j$.

The value $M(i,j)$ will be denoted by m_{ij}. The Coxeter Group $C(M)$ is presented by (I,E) where E is the set of equations $(ij)^{m_{ij}}=1$, $m_{ij}\neq\infty$. As $m_{ii}=1$ implies $i^{-1}=i$, we may represent the elements of $C(M)$ by words from the free monoid I^* on I. If the words w and w' define the same abstract element of $C(M)$ we write $w=_M w'$. Syntactic equality (equality in I^*) is noted as usual $w=w'$.

Throughout this section, $[ij]^k$ will denote the product $ijij\cdots$ of k generators alternatively equal to i and j; α will denote $[ij]^{m_{ij}-1}$. The generators i and j will be denoted by $f(\alpha)$ and $s(\alpha)$ respectively, and $l(\alpha)$ is the last generator of α, equal to i (resp. j) when m_{ij} is even (resp. odd). To α is associated the word $\bar{\alpha}=[ji]^{m_{ij}-1}$. Finally, m_{ij} will be abbreviated in m_α. The same definitions stand for $\beta=[ij]^{m_{ij}}$ and $\gamma=[ij]^{m_{ij}-2}$.

A first solution to the word problem is given by a theorem due to J. Tits (Thm 1, p.93 of [Bou78]). If the generators from I are interpreted by the following linear transformations of a real vector space with basis $e_1,\ldots,e_n$:

$$s_i : e_j \longrightarrow e_j - 2(\cos\frac{\pi}{m_{ij}})e_i$$

then a word $w=i_1\cdots i_k$ from I^* represents the unit element in $C(M)$ iff $s_{i_1}\cdots s_{i_k}(\sum_{j=1}^{n} e_j)=\sum_{j=1}^{n} e_j$. As noted by J. Tits [Tit69], this solution is not efficient. A second solution was proposed in [Tit69] based upon a reduction in $L(I)$ defined by the following rules:

$wiiw' \longrightarrow ww'$, $i\in I$, $w,w'\in I^*$.

$w\beta w' \longrightarrow w\bar{\beta}w'$, $w,w'\in I^*$.

The confluence of this system is proved via the linear representation of Coxeter groups. As the reduction is not length increasing, the enumeration of the words reduced from a given one w halts, we can check whether or not $w=_M 1$. The Knuth-Bendix completion may be used to significantly improve this algorithm. We now detail the completion of a Coxeter group defined by a matrix $M=(m_{ij})$, under a given lexicographic ordering. Together with a constant set of rules, the completion generates new rules sharing a common structure described by a single meta-rule. We restrict ourselves to matrices without entry equal to 2. The completion begins by symmetrizing the given presentation:

Lemma 4
Given a Coxeter matrix M on the set I totally ordered by $>$, the completion generates the two following sets of rules:

$$R_I=\{\, i^{-1}\longrightarrow i,\ ii\longrightarrow 1 \mid i\in I \,\} \text{ and } S_I=\{\, \beta\longrightarrow\bar{\beta} \mid f(\beta)>s(\beta) \,\}.$$

Proof. R_I is generated by the defining relations $ii=1$ and their normal pairs for all i in I. We have $ii>1$, the pairs are (i^{-1},i) and $(i^{-1}i^{-1},1)$. Putting $i^{-1}>i$, the rule $i^{-1}\longrightarrow i$ is generated, under which the second pair is confluent.

S_I is generated by a sequence of normal pairs from the defining rules. If m_{ij} is even, this rule is $\beta\beta\longrightarrow 1$. It generates $\beta\alpha\longrightarrow l(\beta)$. The first rule is redundant and deleted as $\beta\beta$ reduces to $l(\beta)l(\beta)$, then to 1, by the new rule and R_I. This sequence of operations loops and reaches a pair $(\beta,\bar{\beta})$. Then, a rule of type S_I is produced. The case m_{ij} odd is similar ∎

From R_I we can restrict to words in I^*. These rules are length decreasing, while those from S_I shift generators i and j. Note that a word α is both a left member suffix (resp. prefix) and a right member prefix (resp. suffix) of a rule in S_I when $f(\alpha)<s(\alpha)$ (resp. >), and that both $\alpha\alpha$ and $\alpha\overline{\alpha}$ reduce on 1.

Theorem 5
Let M be a Coxeter matrix on the set I totally ordered by >. If $m_{ij}\neq 2$, $i,j\in I$, then the completion procedure generates the set of rules $R_I\cup T_I$, where T_I consists of all rules of the form

$$\alpha_1\cdots\alpha_n l(\overline{\alpha_n})\rightarrow s(\alpha_1)\alpha_1\cdots\alpha_n \tag{T}$$

where n is a positive integer, and for all p such that $1<p<n$:

$$f(\alpha_1)>s(\alpha_1),\ s(\alpha_p)>f(\alpha_p),\ f(\alpha_{p+1})\neq l(\alpha_p),\ s(\alpha_{p+1})=l(\overline{\alpha_p}) \tag{C}$$

A R-rule (resp. S, T) means a rule from R_I (resp. S_I, T_I). The proof can be found in [LeC85]. We just prove by induction that the rules are consequences of the definitions. If $n=1$, then the meta-rule reduces to a S-one. Otherwise $\alpha_n l(\overline{\alpha_n})=s(\alpha_n)\alpha_n$, apply then the induction hypothesis with the last (C) equality.

Let us see an example (when we give a complete set of rules, we shall omit the first ones from R_I). The group is defined on three generators a,b and c with $m_{ab}=4$, $m_{bc}=5$ and $m_{ca}=6$. We give two complete presentations, the first one is defined by the ordering $b>a>c$:

$$G1\left\{\begin{array}{rcl} baba & \rightarrow & abab \\ bcbcb & \rightarrow & cbcbc \\ acacac & \rightarrow & cacaca \\ bcbcabab & \rightarrow & cbcbcaba \\ babcacaca & \rightarrow & ababcacac \\ bcbcabacbcbc & \rightarrow & cbcbcabacbcb \end{array}\right.$$

Each rule is a T one. The complete system has twelve rules. A smaller system is associated with the ordering $a>c>b$:

$$G2\left\{\begin{array}{rcl} abab & \rightarrow & baba \\ cbcbc & \rightarrow & bcbcb \\ acacac & \rightarrow & cacaca \\ acacabcbcb & \rightarrow & cacacabcbc \end{array}\right.$$

Thus, the number of rules depends on the ordering. However, this number does not matter if all rules fall under a single parametrized one. The set of rules may be infinite. Here is an example:

$$G3\left\{\begin{array}{rcl} dcd & \rightarrow & cdc \\ dbdb & \rightarrow & bdbd \\ dada & \rightarrow & adad \end{array}\right.$$

The completion of this set creates infinitely many rules of type $dcbdb(adabdb)^n d\rightarrow cdcbdb(adabdb)^n$, $n\geq 0$. Thus we have examples of infinite sets of identities defining efficient algorithms. Noteworthy, all T-rules are in Post normal form: they can be written as $Va\rightarrow bV$, where V is

$\alpha_1 \cdots \alpha_n$. Before the study of a reduction algorithm, we examine presentations with commuting pairs of generators.

First, let M be a Coxeter matrix with infinite m_{ij}'s, then the meta-rule T always gives the complete system, and Theorem VI.3 is valid for M with the convention that no component α exists for i and j. The meta-rule is puzzled only when some entries of M are equal to 2, i.e. $ij = ji$, the generators commute. This case implies that instances of the meta-rule are S-reducible. With the following *complete* system:

$$G4 \left\{ \begin{array}{rcl} ad & \rightarrow & da \\ bd & \rightarrow & db \\ ca & \rightarrow & ac \\ cd & \rightarrow & dc \\ cbc & \rightarrow & bcb \\ cbac & \rightarrow & bcba \\ cbabcb & \rightarrow & bcbabc \end{array} \right.$$

The T-rule $cbadc \rightarrow bcbad$ is never created as its members are confluent under the commutativity rules and the T-rule $cbac \rightarrow bcba$. Critical pairs are reducible with S-rules, then with arbitrary T-ones. Moreover, some T-rules are partially reduced by the commutativity laws (cf. in last section B_n complete presentations). As final drawback, when a Coxeter matrix has infinite coefficients and others equal to 2, new kinds of rules appear, with:

$$G5 \left\{ \begin{array}{rcl} ca & \rightarrow & ac \\ cb & \rightarrow & bc \\ dad & \rightarrow & ada \\ dbd & \rightarrow & bdb \\ dcd & \rightarrow & cdc \end{array} \right.$$

the completion procedure generates infinitely many rules $dxcd[yx]^n c \rightarrow xdxcd[yx]^n$, $n \geq 0$ where $\{x,y\} = \{a,b\}$. To moderate these negative results, the last section presents some complete systems of Coxeter groups with commuting pairs of generators.

3.2. A Reduction Algorithm

The simplest reduction algorithm iterates the search of a left member, and the substitution of right members. This is of little practical interest. We begin by some remarks on rules and overlapping reductions.

Our goal is a reduction algorithm without backward search in a word already scanned and reduced. After a reduction, what are the possible ones overlapping the new right member ? We first restrict our attention to T-rules:

$$W \xrightarrow{*} v\alpha_1 \cdots \alpha_{n-1}\beta_n w$$
$$\rightarrow v\overline{\beta_1}\alpha_2 \cdots \alpha_n w$$

Any reduction of a v suffix also reduces at most the subword $\overline{\beta_1}$ by the condition C. If it reduces a $\overline{\beta_1}$ prefix of at least two letters, then the same condition C between $\overline{\beta_1}$ and the v suffix implies that $v\alpha_1$ is also reducible. In order to avoid backtracking, the reduction strategy is leftmost, keeping the word v irreducible. If the $\overline{\beta_1}$ prefix is a single generator, reductions may occur, with example G1, we have :

$acaca\ bcbcabab \rightarrow acaca\ \mathbf{c}bcbcaba = acaca\mathbf{c}\ bcbcaba \rightarrow cacacabcbcaba$ (E1)

Thus, the algorithm must update a stack of old left redexes. These prefixes are kept with their occurrence number in the reduced part of the input word. Note that all such prefixes must be stored: we can reach a word vpv_1v_2sw where v_2 is the reverse of v_1, vpv_1 is irreducible and ps is a left member. Then by S,R-reductions, we get the word $vpsw$.

On the right side of the right member, inequalities C imply only one possible reduction: a suffix of the last component α_n is a prefix of the first component of the next redex. But before the reduction we had the configuration $v\alpha_1 \cdots \alpha_n l(\overline{\alpha_n})l(\overline{\alpha_n})w'$. Also, to overcome such overlapping reductions, the algorithm will R-reduce on the right of a T-redex before the T-reduction. Therefore, we may restart the search on the last generator of the right member. With the same example G1:

$bcbcabab\ cacac \rightarrow cbcbcab\mathbf{a}\ cacac = cbcbcab\ \mathbf{a}cacac \rightarrow bcbcabcacaca$ (E2)

The two facts that 1) the new generator (c in E1) can only be the last one of a new redex just before the old one and 2) we can skip until the last generator of the right member (a in E2) are the basic points of the reduction algorithm.

Let us now look more closely to the possible R-reductions following a T-one. On the left, we claim that at most one R-reduction can occur. Otherwise, the T-redex would not be the leftmost one as $f(\alpha_1)s(\alpha_1)\alpha_1 \rightarrow \overline{\beta_1}l(\alpha_1)$. With example G1:

$$bc\ bcbcabab \rightarrow bc\ cbcbcaba \xrightarrow{*} cbcaba$$

But $bc\ bcbcabab = bcbcb\ cabab$, and the redex $bcbcabab$ is not the leftmost one. On the right hand of a right member, we may of course have several R-reductions. Here we may observe that the leftmost strategy is also more efficient than the rightmost one:

$$bcbcabab\ a \rightarrow cbcbcaba\ a \rightarrow cbcbcab$$

While $bcbca\ baba \rightarrow bcbca\ abab \rightarrow bcbcbab \rightarrow cbcbcab$

The rightmost strategy induces a shift/reduce process wich makes $n+1$ reductions while the leftmost one always produce two reductions, n the number of components in the leftmost redex.

It remains to examine the consequences of R-reductions. On another R-reduction, they are taken in account by a loop deleting the common generators at top and bot of the unscanned and reduced part of the input word. On a T-rule, a R-reduction may increase the last redex prefix at top of the redex stack. Thus a sequence of R-reductions must be closed by an update. This update splits in two operations: the removal of prefixes deleted by R-reductions, and possibly a pop operation on the stack, so that its top becomes the current prefix.

These two observations outline the global strategy of an efficient reduction algorithm based on a leftmost strategy. Let us present more formally the main iteration of the algorithm. This loop updates four variables: v the reduced part of the initial word, w the remaining input, r the current T-redex prefix, and s, the stack: list of redex prefixes together with their occurrence in v. Entering the loop, the word is equal to vrw, where $r=\alpha_1 \cdots \alpha_k$, C being satisfied. Building r needs a function *component* which recognizes a component in the word w, i.e. $w=\alpha k w'$, where $k \neq l(\alpha)$. Also a boolean function *link* returns true if condition C between the last component α_k of r and α is satisfied, observe that this function uses the assumption $m_{ij} \neq 2$. A procedure

apply applies the rule whose left member has just been recognized, this function possibly pops the stack as a R-rule may appear on the left of the redex right member. The update of r, v and the stack s after a R-reduction is performed by a procedure ***update***. Finally, ***plast*** returns the last generator of $\overline{\alpha}$, and, given two generators a,b, ***alpha*** computes α_{ab}. A list is noted [a;b;c], the dot . and the at @ are the list cons and append, ***last*** and ***tail*** are usual list functions.

Main loop of the reduction algorithm

```
loop {
While w=aaw'  do { w:=w' };
While w=abbw' do { w:=aw' };
If a=last(v@r) Then { w:=tail(w); update(v,r,s) }
Else {
(bool, w', c, d):=component(a,b,(tail (tail w)));
If bool Then  {                  (w starts with a component)
  If a>b Then { If c=plast(a,b) Then { apply([],a,b,(d.w'),(v@r)) }
                Else { v:=v@r; push(s,r); r:=alpha(a b) }}
  Else { If r=[] Then { v:=v@alpha(a,b) }
         Else {  If c=plast(a,b) and link(a,b,r) Then { apply(r,a,b,(d.w'),v) }
                 Else { r:=r@alpha(a,b) }}}}
Else { v:=v@[a;b]; w:=w'; push(s,r); r:=[] }}} ■
```

3.3. Some Examples

In all examples, the set R_I is omitted. We consider finite Coxeter groups first described by H.S.M Coxeter [Cox35]. The notations are taken from [Bou78]. The completion of a finite group always halts [LeC85]. The finite Coxeter groups whose matrix entries are equal to 1,2,3,4 or 6 are called crystallographic groups. However we failed to complete the following crystallographic groups: E_n, n=6,7,8 and the family D_n.

Groups H_4, H_3 and $I_2(n)$.

$$H_4 \left\{ \begin{array}{r@{\;\longrightarrow\;}l} dcd & cdc \\ cbc & bcb \\ cbac & bcba \\ babab & ababa \\ cbabcb & bcbabc \\ cbabacbaba & bcbabacbab \end{array} \right. \qquad H_3 \left\{ \begin{array}{r@{\;\longrightarrow\;}l} cbc & bcb \\ cbac & bcba \\ babab & ababa \\ cbabcb & bcbabc \\ cbabacbaba & bcbabacbab \end{array} \right.$$

These two groups are not crystallographic, nor are the dihedral groups $I_2(n)$, n>4, except $I_2(6)$. These groups are generated by two plane reflections through lines whose angle is $\frac{2\pi}{n}$, their complete presentation is the simpler one, all critical pairs are solved by symmetrization, the complete set is $R_I \cup S_I$. The remaining finite groups are crystallographic.

Groups F_4, E_6, E_7 and E_8.

$$F_4 \left\{ \begin{array}{r@{\;\longrightarrow\;}l} bab & aba \\ dcd & cdc \\ cbcb & bcbc \\ cbacba & bcbacb \end{array} \right.$$

The Knuth-Bendix procedure failed to complete the three groups E_n, n=6,7,8 while generating hundred rules such as, for E_6:

$$fcbadcbedfcbadc \rightarrow cfcbadcbedfcbad$$
$$fcbadcbedcfcbdcf \rightarrow cfcbadcbedcfcbdc$$

As these groups possess many matrix entries equal to 2, we do not have an efficient reduction algorithm for these three groups.

Groups A_n, B_{n+1} and D_{n+3}, n≥1.

These three families define the infinite families of crystallographic groups. The first family is the family of symmetric groups. Their complete presentation may be found in the last section. Despite commuting pairs of generators, observe that we have only T-rules. The groups B_n also possess a fair complete presentation. Their Coxeter matrix is

$$\begin{bmatrix} 1 & 3 & & & \mathbf{2} & \\ 3 & 1 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 1 & 3 & 2 \\ & \mathbf{2} & & 3 & 1 & 4 \\ & & & 2 & 4 & 1 \end{bmatrix}$$

The complete system includes the rules R_I, the rules of commutativity, and the following T-rules for B_n :

$$\begin{cases} a_i a_{i-1} \cdots a_{i-k} a_i \rightarrow a_{i-1} a_i a_{i-1} \cdots a_{i-k} & n>i>k>0 \\ (a_n a_{n-1} \cdots a_{n-k})^2 \rightarrow a_{n-1} a_n a_{n-1} \cdots a_{n-k} a_n a_{n-1} \cdots a_{n-k+1} & n>k>0 \end{cases}$$

The last family D_n however does not possess an easily described complete system. Their Coxeter matrix is the previous one where the last row and column are replaced by [2 2 ... 2 3 2 1]. Here are the T-rules except the commutative ones for D_4:

$$\left\{ \begin{array}{lcl lcl} dbd & \rightarrow & bdb & dbad & \rightarrow & bdba \\ cbc & \rightarrow & bcb & dbcd & \rightarrow & bdbc \\ bab & \rightarrow & aba & dbacd & \rightarrow & bdbac \\ cbac & \rightarrow & bcba & dbacba & \rightarrow & cdbacb \\ dbcb & \rightarrow & cdbc & dbacbdb & \rightarrow & bdbacbd \end{array} \right.$$

4. Polyhedral Groups

The polyhedral group (l,m,n) is defined by the presentation $(A,B,C\ ;\ ABC,A^l,B^m,C^n)$ [Cox72]. We present in this section complete systems for the following generalization:

$$(p_1, \ldots, p_n) = (A_1, \ldots, A_n; A_1^{p_1}, \ldots, A_n^{p_n}, A_1 \cdots A_n) \quad n>2.$$

Observe that these groups are subgroups of Coxeter groups (the rotation subgroups). As for Coxeter groups, the general complete system requires $p_i>2$, $i=1, \ldots, n$. Let us first examine the case n=3. This presentation is redundant, one of the generators, say C, may be eliminated. Then we can see on the new presentation $(A,B;A^l,B^m,(AB)^n)$ that when l,m and n are greater than 3 the group is a small cancellation one, as the only pieces are the generators and their inverses. Thus its word problem is solvable. The groups are infinite when $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\le 1$. Thus, the only finite ones are $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. The first one is the group of rotations and reflections of a regular p-gon of the euclidian plane. Its complete system is quite trivial:

$$(2,2,n)\left\{\begin{array}{lcl} A^{-1} & \to & A \\ B^{-1} & \to & B \\ AA & \to & 1 \\ BB & \to & 1 \end{array}\right.$$

and, if $n=2p+1$ $(AB)^p A \to (BA)^p B$

if $n=2p$ $(AB)^p \to (BA)^p$

The remaining finite groups are the rotation groups of the five regular polyhedrons of the three dimensional space. As isolated groups they possess many complete presentations (recall that for finite groups the completion always halts). The essential point with complete systems is of course to obtain parametrized ones while keeping a (relatively) terse description.

Tetraedron:

$$\left\{\begin{array}{lcl lcl} C^{-1} & \to & CC & CCC & \to & 1 \\ B^{-1} & \to & BB & BBB & \to & 1 \\ CBC & \to & BB & BCB & \to & CC \\ CCBB & \to & BC & BBCC & \to & CB \\ CBBC & \to & BCCB & & & \end{array}\right.$$

Termination: KB Ordering with $\pi(B^{-1})=3$, $\pi(C^{-1})=3$, $\pi(B)=\pi(C)=1$ and $C>B$ for the last equation.

Cube (or its dual **Octaedron**):

$$\left\{\begin{array}{rcl} A^{-1} & \to & A \\ C^{-1} & \to & ACACA \\ AA & \to & 1 \\ CCC & \to & ACACA \\ CACAC & \to & A \\ CACCAC & \to & ACCA \\ CCACCA & \to & ACCACC \end{array}\right.$$

Termination: KB Ordering with $\pi(C^{-1})=5, \pi(A^{-1})=\pi(C)=\pi(A)=1$, and $C^{-1}>A^{-1}>C>A$.

Icosaedron (or **Dodecaedron**):

$$\left\{\begin{array}{rcl rcl} A^{-1} & \to & A & AA & \to & 1 \\ B^{-1} & \to & BB & BBB & \to & 1 \\ BABABAB & \to & ABBA & BBABB & \to & ABABABA \\ BABABBABAB & \to & ABBABABBA & BABBABABBABA & \to & ABABBABABBAB \end{array}\right.$$

Termination: KB Ordering with $\pi(B^{-1})=6, \pi(B)=3, \pi(A^{-1})=\pi(A)=1$, and $A^{-1}>B^{-1}>A>B$.

The remaining groups are infinite, we may suppose that $l\le m\le n$, because of the symmetry of the presentation (all the groups (l,m,n) and (p,q,r) with $\{l,m,n\}=\{p,q,r\}$ are isomorphic). There are two distinct cases, either two is the power of some generator or not. We give the general system when no parameter equals two. This set of rules divides in three groups of six rules. The simpler case is when all the parameters are odd: $l=2p+1, m=2q+1, n=2r+1$. The remaining cases have slight modifications of the exponents. The first set is

$$\begin{cases} AB \rightarrow C^{-1} \\ BC \rightarrow A^{-1} \\ CA \rightarrow B^{-1} \\ A^{-1}C^{-1} \rightarrow B \\ C^{-1}B^{-1} \rightarrow A \\ B^{-1}A^{-1} \rightarrow C \end{cases}$$

This is the symmetrized set of the defining relation ABC. Then we have six symmetrized rules of the remaining relations.

$$\begin{cases} A^{p+1} \rightarrow A^{-p} \\ A^{-(p+1)} \rightarrow A^{p} \\ B^{q+1} \rightarrow B^{-p} \\ B^{-(q+1)} \rightarrow B^{p} \\ C^{r+1} \rightarrow C^{-r} \\ C^{-(r+1)} \rightarrow C^{r} \end{cases}$$

And finally, we have critical pairs rules:

$$\begin{cases} A^{-1}C^{r} \rightarrow BC^{-r} \\ A^{-p}B \rightarrow A^{p}C^{-1} \\ B^{-1}A^{p} \rightarrow CA^{-p} \\ B^{-q}C \rightarrow B^{q}A^{-1} \\ C^{-1}B^{q} \rightarrow AB^{-q} \\ C^{-r}A \rightarrow C^{r}B^{-1} \end{cases}$$

Termination: Lex Ordering with $C^{-1}>B^{-1}>A^{-1}>C>B>A$.

Three cases remain, when one, two or three generators have an even order. The set (1) of rewrite rules remains the same. The set (2) is modified when the order of a generator becomes even, say A has order $2p$, then the two corresponding rules are:

$$\begin{cases} A^{p+1} \rightarrow A^{-(p-1)} \\ A^{-p} \rightarrow A^{p} \end{cases}$$

Then, in the last set of rules (3), we now have two modifications:

$$\begin{cases} A^{-(p-1)}B \rightarrow A^{p}C^{-1} \\ B^{-1}A^{p} \rightarrow CA^{-(p-1)} \end{cases}$$

These modifications occur for every generator, whatever is the order of the parity shift. For example we give four complete sets (7,7,7), (7,8,9), (7,8,8) and (8,8,8).

$$\left\{\begin{array}{rcl} AB & \rightarrow & C^{-1} \\ BC & \rightarrow & A^{-1} \\ CA & \rightarrow & B^{-1} \\ A^{-1}C^{-1} & \rightarrow & B \\ C^{-1}B^{-1} & \rightarrow & A \\ B^{-1}A^{-1} & \rightarrow & C \\ A^{4} & \rightarrow & A^{-3} \\ A^{-4} & \rightarrow & A^{3} \\ B^{4} & \rightarrow & B^{-3} \\ B^{-4} & \rightarrow & B^{3} \\ C^{4} & \rightarrow & C^{-3} \\ C^{-4} & \rightarrow & C^{3} \\ B^{-1}A^{3} & \rightarrow & CA^{-3} \\ A^{-3}B & \rightarrow & A^{3}C^{-1} \\ C^{-1}B^{3} & \rightarrow & AB^{-3} \\ B^{-3}C & \rightarrow & B^{3}A^{-1} \\ A^{-1}C^{3} & \rightarrow & BC^{-3} \\ C^{-3}A & \rightarrow & C^{3}B^{-1} \end{array}\right\} \qquad \left\{\begin{array}{rcl} AB & \rightarrow & C^{-1} \\ BC & \rightarrow & A^{-1} \\ CA & \rightarrow & B^{-1} \\ A^{-1}C^{-1} & \rightarrow & B \\ C^{-1}B^{-1} & \rightarrow & A \\ B^{-1}A^{-1} & \rightarrow & C \\ A^{4} & \rightarrow & A^{-3} \\ A^{-4} & \rightarrow & A^{3} \\ B^{5} & \rightarrow & B^{-3} \\ B^{-4} & \rightarrow & B^{4} \\ C^{5} & \rightarrow & C^{-4} \\ C^{-5} & \rightarrow & C^{4} \\ B^{-1}A^{3} & \rightarrow & CA^{-3} \\ A^{-3}B & \rightarrow & A^{3}C^{-1} \\ C^{-1}B^{4} & \rightarrow & AB^{-3} \\ B^{-3}C & \rightarrow & B^{4}A^{-1} \\ A^{-1}C^{4} & \rightarrow & BC^{-4} \\ C^{-4}A & \rightarrow & C^{4}B^{-1} \end{array}\right\}$$

$$\left\{\begin{array}{rcl} AB & \rightarrow & C^{-1} \\ BC & \rightarrow & A^{-1} \\ CA & \rightarrow & B^{-1} \\ A^{-1}C^{-1} & \rightarrow & B \\ C^{-1}B^{-1} & \rightarrow & A \\ B^{-1}A^{-1} & \rightarrow & C \\ A^{4} & \rightarrow & A^{-3} \\ A^{-4} & \rightarrow & A^{3} \\ B^{5} & \rightarrow & B^{-3} \\ B^{-4} & \rightarrow & B^{4} \\ C^{5} & \rightarrow & C^{-3} \\ C^{-4} & \rightarrow & C^{4} \\ B^{-1}A^{3} & \rightarrow & CA^{-3} \\ A^{-3}B & \rightarrow & A^{3}C^{-1} \\ C^{-1}B^{4} & \rightarrow & AB^{-3} \\ B^{-3}C & \rightarrow & B^{4}A^{-1} \\ A^{-1}C^{4} & \rightarrow & BC^{-3} \\ C^{-3}A & \rightarrow & C^{4}B^{-1} \end{array}\right\} \qquad \left\{\begin{array}{rcl} AB & \rightarrow & C^{-1} \\ BC & \rightarrow & A^{-1} \\ CA & \rightarrow & B^{-1} \\ A^{-1}C^{-1} & \rightarrow & B \\ C^{-1}B^{-1} & \rightarrow & A \\ B^{-1}A^{-1} & \rightarrow & C \\ A^{5} & \rightarrow & A^{-3} \\ A^{-4} & \rightarrow & A^{4} \\ B^{5} & \rightarrow & B^{-3} \\ B^{-4} & \rightarrow & B^{4} \\ C^{5} & \rightarrow & C^{-3} \\ C^{-4} & \rightarrow & C^{4} \\ B^{-1}A^{4} & \rightarrow & CA^{-3} \\ A^{-3}B & \rightarrow & A^{4}C^{-1} \\ C^{-1}B^{4} & \rightarrow & AB^{-3} \\ B^{-3}C & \rightarrow & B^{4}A^{-1} \\ A^{-1}C^{4} & \rightarrow & BC^{-3} \\ C^{-3}A & \rightarrow & C^{4}B^{-1} \end{array}\right\}$$

The termination is now a hard problem. At least one, and at most three rules in the even case are length increasing, and no classical ordering proves the noetherianity. From hand-made examples, we conjecture that the reductions are noetherian.

The irreducible forms of (l,m,n) are described by the finite automaton of figure 3 (recall that the set of normal forms is regular), with the following conventions:

— A state labeled A (resp. B,C) recognizes the subwords A^{i}, $i=1,\ldots,[l/2]$. A state labeled a (resp. b,c) recognizes the subwords A^{-i}, $i=1,\ldots,[(l-1)/2]$.

— Simple arrows allow all transitions, whatever is the subword recognized by the initial state of the arrow. Double arrows allow all transitions but the one whose initial state has recognized the maximal length subword (rules with left members $B^{-3}C$). Triple arrows allow all transitions but the one whose final state recognizes the maximal subword (rules with left members

$B^{-1}A^3$).

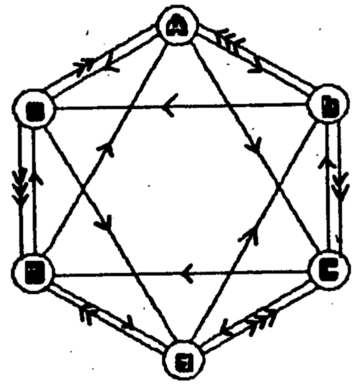

Fig. 3

We now describe the complete systems of "polyhedral" groups defined by at least four generators. We have two cases according to the parity of the number of generators. As for Coxeter groups, we restrict ourselves to generators of period greater than 2.

4.1. Polyhedral groups with odd number of generators

Let $G=\{A_1,\ldots,A_{2n+1}\}$ be the linearly ordered set of generators. Let $\alpha_1\cdots\alpha_{n+1}$ be any subword of length $n+1$ in the word $W_G=A_1\cdots A_{2n+1}A_1\cdots A_n$.
The word $\alpha_{n+2}\cdots\alpha_{2n+1}$ denotes its *complement*: suffix of length n, or prefix of length n if such a suffix does not exist. We give the complete presentation for odd exponents. The complete system for $(2p_1+1,\ldots,2p_{2n+1}+1)$, $p_i>0$, is therefore:

$$\begin{array}{rcl}
\alpha_1\cdots\alpha_{n+1} & \to & (\alpha_{n+2}\cdots\alpha_{2n+1})^{-1} \\
(\alpha_1\cdots\alpha_{n+1})^{-1} & \to & \alpha_{n+2}\cdots\alpha_{2n+1} \\
\alpha^{p_\alpha+1} & \to & \alpha^{-p_\alpha} \\
\alpha^{-(p_\alpha+1)} & \to & \alpha^{p_\alpha} \\
\alpha_{n+1}^{-1}\cdots\alpha_2^{-1}\alpha_1^{p_{\alpha_1}} & \to & \alpha_{n+2}\cdots\alpha_{2n+1}\alpha_1^{-p_{\alpha_1}} \\
\alpha^{-p_{\alpha_1}}\alpha_2\cdots\alpha_{n+1} & \to & \alpha_1^{p_{\alpha_1}}(\alpha_{n+2}\cdots\alpha_{2n+1})^{-1}
\end{array}$$

For complete presentations with generators α of even order $2p_\alpha$, the third and fourth rules become $\alpha^{p_\alpha+1}\to\alpha^{-(p_\alpha-1)}$ and $\alpha^{-p_\alpha}\to\alpha^{p_\alpha}$ respectively. And the other pairs of exponents $(p_\alpha,-p_\alpha)$ become $(p_\alpha,-(p_\alpha-1))$. Here is for example the rules for (6,5,5,5,5):

$ABC\to E^{-1}D^{-1}$	$BCD\to A^{-1}E^{-1}$	$CDE\to B^{-1}A^{-1}$	$DEA\to C^{-1}B^{-1}$	$EAB\to D^{-1}C^{-1}$
$A^{-1}E^{-1}D^{-1}\to BC$	$B^{-1}A^{-1}E^{-1}\to CD$	$C^{-1}B^{-1}A^{-1}\to DE$	$D^{-1}C^{-1}B^{-1}\to EA$	$E^{-1}D^{-1}C^{-1}\to AB$
$AAAA\to A^{-1}A^{-1}$	$BBB\to B^{-1}B^{-1}$	$CCC\to C^{-1}C^{-1}$	$DDD\to D^{-1}D^{-1}$	$EEE\to E^{-1}E^{-1}$
$A^{-1}A^{-1}A^{-1}\to AAA$	$B^{-1}B^{-1}B^{-1}\to BB$	$C^{-1}C^{-1}C^{-1}\to CC$	$D^{-1}D^{-1}D^{-1}\to DD$	$E^{-1}E^{-1}E^{-1}\to EE$
$A^{-1}E^{-1}DD\to BCD^{-1}D^{-1}$	$B^{-1}A^{-1}EE\to CDE^{-1}E^{-1}$	$C^{-1}B^{-1}AAA\to DEA^{-1}A^{-1}$	$D^{-1}C^{-1}BB\to EAB^{-1}B^{-1}$	$E^{-1}D^{-1}CC\to ABC^{-1}C^{-1}$
$A^{-1}A^{-1}BC\to AAAE^{-1}D^{-1}$	$B^{-1}B^{-1}CD\to BBA^{-1}E^{-1}$	$C^{-1}C^{-1}DE\to CCB^{-1}A^{-1}$	$D^{-1}D^{-1}EA\to DDC^{-1}B^{-1}$	$E^{-1}E^{-1}AB\to EED^{-1}C^{-1}$

The number of rules is $6|G|$. As for surface groups, the sets of rules are simply described geometrically. We display the rules in the Cayley graph, starting at a given vertex. We present (5,5,5). For the other groups with odd number $2n+1$ of generators the number of polygons around the central vertex is

$4n+2$. The grey regions denote the defining relation $ABC=1$. The Cayley graphs are planar, so that they are oriented according to the displayed arrow. The size of the other polygons depends on the order of the generator. Fat lines denotes forbidden edges for paths in normal form.

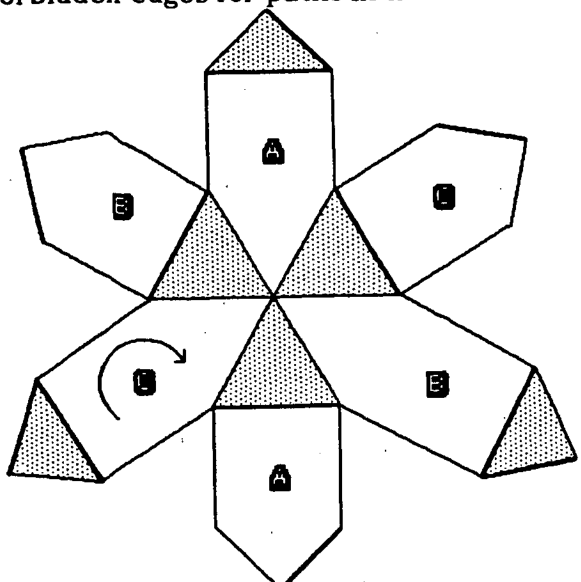

Fig. 4, (5,5,5).

4.2. Polyhedral groups with even number of generators

The number of rules is now $10|G|$, instead of $6|G|$. Let $2n$ be the number of generators and put $W_G=A_1\cdots A_{2n}A_1\cdots A_n$. The words $\alpha_1\cdots\alpha_{n+1}$ have the same definition as above. We also need the words $\alpha_1\cdots\alpha_n$ together with their complements $\alpha_{n+1}\cdots\alpha_{2n}$. We give the complete system for generators of odd exponent. The complete system for $(2p_1+1,\ldots,2p_{2n}+1)$, $p_i>0$, is:

$$\alpha_1\cdots\alpha_{n+1}\longrightarrow(\alpha_{n+2}\cdots\alpha_{2n})^{-1}$$
$$(\alpha_1\cdots\alpha_n)^{-1}\longrightarrow\alpha_{n+1}\cdots\alpha_{2n}$$
$$\alpha^{p_\alpha+1}\longrightarrow\alpha^{-p_\alpha}$$
$$\alpha^{-(p_\alpha+1)}\longrightarrow\alpha^{p_\alpha}$$
$$\alpha_1\cdots\alpha_n\alpha_{n+1}^{-p_{\alpha_{n+1}}}\longrightarrow(\alpha_{n+2}\cdots\alpha_{2n})^{-1}\alpha_{n+1}^{p_{\alpha_{n+1}}}$$
$$\alpha^{-p_{\alpha_1}}\alpha_2\cdots\alpha_{n+1}\longrightarrow\alpha_1^{p_{\alpha_1}}(\alpha_{n+2}\cdots\alpha_{2n})^{-1}$$
$$\alpha_1^{-p_{\alpha_1}}\alpha_2\cdots\alpha_n\alpha_{n+1}^{-p_{\alpha_{n+1}}}\longrightarrow\alpha_1^{p_{\alpha_1}}(\alpha_{n+2}\cdots\alpha_{2n})^{-1}\alpha_{n+1}^{p_{\alpha_{n+1}}}$$
$$(\alpha_{n+2}\cdots\alpha_{2n})^{-1}\alpha_{n+1}^{p_{\alpha_{n+1}}}(\alpha_2\cdots\alpha_n)^{-1}\alpha_1^{p_{\alpha_1}}\longrightarrow\alpha_1\cdots\alpha_n\alpha_{n+1}^{-(p_{\alpha_{n+1}}-1)}\alpha_{n+2}\cdots\alpha_{2n}\alpha_1^{-p_{\alpha_1}}$$
$$\alpha_1\cdots\alpha_n\alpha_{n+1}^{-(p_{\alpha_{n+1}}-1)}\alpha_{n+2}\cdots\alpha_{2n}\alpha_1\longrightarrow(\alpha_{n+2}\cdots\alpha_{2n})^{-1}\alpha_{n+1}^{p_{\alpha_{n+1}}}(\alpha_2\cdots\alpha_n)^{-1}$$
$$\alpha_1^{-p_{\alpha_1}}\alpha_2\cdots\alpha_n\alpha_{n+1}^{-(p_{\alpha_{n+1}}-1)}\alpha_{n+2}\cdots\alpha_{2n}\alpha_1\longrightarrow\alpha_1^{p_{\alpha_1}}(\alpha_{n+2}\cdots\alpha_{2n})^{-1}\alpha_{n+1}^{p_{\alpha_{n+1}}}(\alpha_2\cdots\alpha_n)^{-1}$$

For generators with even exponent, the observations of the previous section remain valid, together with the convention that $-(p_\alpha-1)$ becomes $-(p_\alpha-2)$ for a generator α of exponent $2p_\alpha$. We give for example (6,5,5,5):

$ABC \to D^{-1}$
$A^{-1}D^{-1} \to BC$
$AAAA \to A^{-1}A^{-1}$
$A^{-1}A^{-1}A^{-1} \to AAA$
$ABC^{-1}C^{-1} \to D^{-1}CC$
$A^{-1}A^{-1}BC^{-1}C^{-1} \to AAAD^{-1}CC$
$ABC^{-1}DA \to D^{-1}CCB^{-1}$
$A^{-1}DDC^{-1}BB \to BCD^{-1}AB^{-1}B^{-1}$
$C^{-1}BBA^{-1}DD \to DAB^{-1}CD^{-1}D^{-1}$
$A^{-1}A^{-1}BC^{-1}DA \to AAAD^{-1}CCB^{-1}$
$C^{-1}C^{-1}DA^{-1}BC \to CCB^{-1}AAAD^{-1}$

$BCD \to A^{-1}$
$B^{-1}A^{-1} \to CD$
$BBB \to B^{-1}B^{-1}$
$B^{-1}B^{-1}B^{-1} \to BB$
$BCD^{-1}D^{-1} \to A^{-1}DD$
$B^{-1}B^{-1}CD^{-1}D^{-1} \to BBA^{-1}DD$
$BCD^{-1}AB \to A^{-1}DDC^{-1}$

$CDA \to B^{-1}$
$C^{-1}B^{-1} \to DA$
$CCC \to C^{-1}C^{-1}$
$C^{-1}C^{-1}C^{-1} \to CC$
$CDA^{-1}A^{-1} \to B^{-1}AAA$
$C^{-1}C^{-1}DA^{-1}A^{-1} \to CCB^{-1}AAA$
$CDA^{-1}BC \to B^{-1}AAAD^{-1}$
$B^{-1}AAAD^{-1}CC \to CDA^{-1}BC^{-1}C^{-1}$
$D^{-1}CCB^{-1}AAA \to ABC^{-1}DA^{-1}A^{-1}$
$B^{-1}B^{-1}CD^{-1}AB \to BBA^{-1}DDC^{-1}$
$D^{-1}D^{-1}AB^{-1}CD \to DDC^{-1}BBA^{-1}$

$DAB \to C^{-1}$
$D^{-1}C^{-1} \to AB$
$DDD \to D^{-1}D^{-1}$
$D^{-1}D^{-1}D^{-1} \to DD$
$DAB^{-1}B^{-1} \to C^{-1}BB$
$D^{-1}D^{-1}AB^{-1}B^{-1} \to DDC^{-1}BB$
$DAB^{-1}CD \to C^{-1}BBA$

As in the previous section, we give a geometrical interpretation of the rules, here for (5,5,5,5):

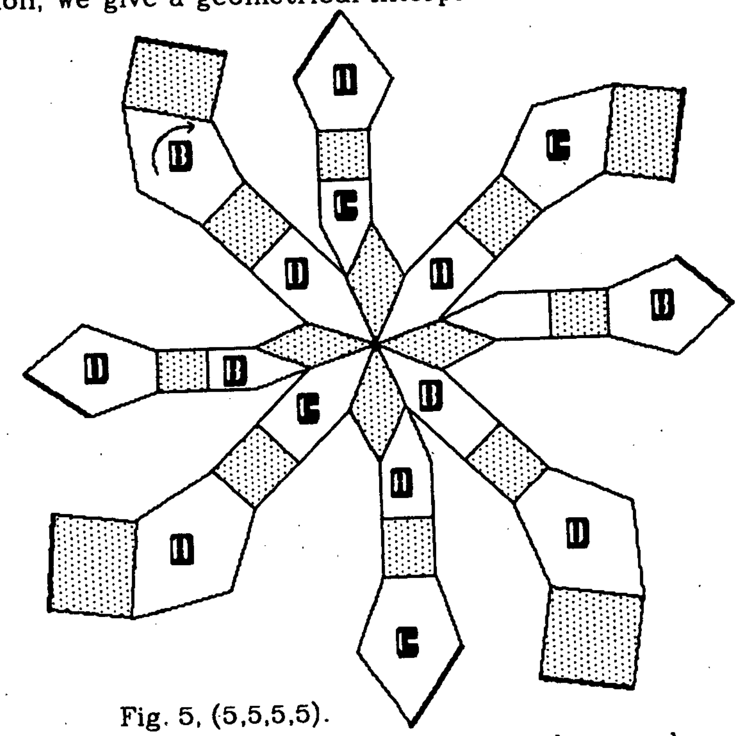

Fig. 5, (5,5,5,5).

When the number of generators increase, so does the number of branches around the central vertex. And the number of edges in polygons varies as the exponent of generators. Observe that the initial cycles surrounding the central vertex give two rules, the others only one. These figures give a concise construction of Cayley graphs. The critical pairs are computed by superposition on a *single* generator, as for Coxeter groups. The completion procedure stops as the remaining superposition creates a subgraph yet appearing somewhere else in the graph. Last, recall that for presentations including generators with even period, we did not prove the termination of the system due to length-increasing rules, such as $A^{-1}A^{-1}BC^{-1}DA \to AAAD^{-1}CCB^{-1}$ in (6,5,5,5).

5. Symmetric Groups

The symmetric group $\mathbf{S}_n$ of order n is the group of all permutations on n objects. We give two complete presentations of the symmetric group.

5.1. Presentation with Adjacent Transpositions

The presentation $\mathbf{S}_n$ by adjacent transpositions is the following one:

$$\begin{cases} R_i &= (i\ i{+}1) & i=1,\dots,n-1 \\ R_i^2 &= 1 & i=1,\dots,n-1 \\ R_i R_j &= R_j R_i & i \le j-2 \\ (R_i R_{i+1})^3 &= 1 & i \le n-2 \end{cases}$$

The completion gives n^2-2n+2 rules:

$$\begin{cases} R_i^{-1} & \to & R_i & i=1,\dots,n \\ R_i^2 & \to & 1 & i=1,\dots,n \\ R_i R_j & \to & R_j R_i & j \le i-2 \\ R_i R_{i-1} \cdots R_j R_i & \to & R_{i-1} R_i R_{i-1} \cdots R_j & j<i \end{cases}$$

Let $1=R_0$ in S_n, then for each rule the integer made by the concatenation of the left members generators indices is greater than the right member one, thus the system is noetherian. The whole set of rules for S_5 is:

$$\begin{cases} R_1^{-1} & \to & R_1 \\ R_2^{-1} & \to & R_2 \\ R_3^{-1} & \to & R_3 \\ R_4^{-1} & \to & R_4 \\ R_1R_1 & \to & 1 \\ R_2R_2 & \to & 1 \\ R_3R_3 & \to & 1 \\ R_4R_4 & \to & 1 \\ R_3R_1 & \to & R_1R_3 \\ R_4R_1 & \to & R_1R_4 \\ R_2R_4 & \to & R_2R_4 \\ R_2R_1R_2 & \to & R_1R_2R_1 \\ R_3R_2R_3 & \to & R_2R_3R_2 \\ R_4R_3R_4 & \to & R_3R_4R_3 \\ R_3R_2R_1R_3 & \to & R_2R_3R_2R_1 \\ R_4R_3R_2R_4 & \to & R_3R_4R_3R_2 \\ R_4R_3R_2R_1R_4 & \to & R_3R_4R_3R_2R_1 \end{cases}$$

A remarkable feature of the systems S_n is that $S_n \subset S_{n+1}$. Thus the infinite set of rules $S_\infty = \bigcup_{n=1}^{\infty} S_n$ defines a canonical form of a permutation of arbitrary length. As for the surfaces fundamental groups, such a system must be compiled into three efficient algorithms:

- An algorithm of normalization, computing the relation $\xrightarrow[S_\infty]{*}$, using knowledge about the special form of the rules.
- An algorithm to perform the product of two permutations already in irreducible form, for which we know the localization of the possible reductions.

— A computation of P^{-1} from the normal form of P. Moreover, a complete presentation of symmetric groups gives a sorting algorithm. Given a permutation φ as an unsorted list, the normal form of φ^{-1} sorts φ. The reader may check that the above presentation defines insertion sorting.

5.2. Presentation with all Transpositions

We put $T_{i,j}$=(i j) with 1≤i<j≤n. These new generators are related with the previous ones by:

$$T_{i,j}=R_i R_{i+1} \cdots R_{j-2}R_{j-1}R_{j-2}R_{j-3} \cdots R_i$$

The definition of $\mathbf{S}_n$ is therefore:

$$\begin{cases} T_{i,j}^2 & = & 1 & \\ T_{i,j}T_{i,i+1} & = & T_{i,i+1}T_{i+1,j} & \\ (T_{i,i+1}T_{i+1,i+2})^3 & = & 1 & \\ (T_{i,i+1}T_{j,j+1})^2 & = & 1 & \text{i+1<j} \\ T_{i,i+1}T_{i+1,j}T_{i,i+1} & = & T_{i,j} & \text{i+1<j} \end{cases}$$

A possible completion is:

$$\begin{cases} T_{i,j}^{-1} & \rightarrow & T_{i,j} & \\ T_{i,j}T_{i,j} & \rightarrow & 1 & \\ T_{i,j}T_{k,l} & \rightarrow & T_{k,l}T_{i,j} & \text{i}\neq\text{k,i}\neq\text{l,j}\neq\text{l} \\ T_{i,j}T_{i,k} & \rightarrow & T_{i,k}T_{k,j} & \text{i<k<j} \\ T_{i,j}T_{k,j} & \rightarrow & T_{k,i}T_{i,j} & \text{k<i<j} \\ T_{i,j}T_{k,i} & \rightarrow & T_{k,i}T_{k,j} & \text{k<i<j} \end{cases}$$

Termination: Lex Ordering, with all the inverses greater than their corresponding generators and

$$T_{n-1,n}>T_{n-2,n}> \cdots >T_{1,n}>T_{n-2,n-1}>T_{n-3,n-1}> \cdots >T_{1,n-1}> \cdots >T_{1,2}.$$

The number of rules is $O(n^4)$, which is far from the upper bound we gave in chapter IV, here exponential: $(n-1)(n-2)n!$. Once more, we have $T_n \subset T_{n+1}$. Thus $T_\infty = \bigcup_{n=1}^{\infty} T_n$ reduces an arbitrary length permutation to a canonical form. Another prominent feature of this set is that it is a *symmetrized* set. Moreover, the rules enumerates all the quasi-commutativity laws between the transpositions, and these rules are sufficient to compute in $\mathbf{S}_n$. This presentation defines max sorting.

To conclude, we may briefly compare the Todd-Coxeter coset enumeration [Tod36] and the Knuth-Bendix procedure for finite groups. The coset enumeration computes a representation of the Cayley graph, while the completion, by a computation on its cycles, determines a unique path between two vertices. It is therefore obvious that, as quoted by Gilman [Gil79], the coset enumeration is generally more efficient (cf. [Can73] for a detailed analysis of this algorithm). For the group E_6, M.F. Newman (private communication) reports that the Canberra implementation of coset enumeration produced a full coset table in less than three minutes, while we could not complete this group. The main advantages of completion technique in groups is its ability to handle parametrized classes, providing efficient word problems after an analysis of the canonical system. Moreover, the study of Coxeter groups has shown that infinite sets of rules could be described (see also [Ped84] as example of using infinite set of rules in solving the free word problem for the groupoid variety $(x.xy)x=y$). Therefore, these two algorithms appear to be complementary, one being well-suited for isolated groups, the other for parametrized families.

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Imprimé en France

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